Logs

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1 Logarithms

Logarithms allow us to move between multiplicative equations and additive ones.

Logs are defined relative to a base number. If we take 10 as the base then $ y = $\log_{10}$(x)$ means $10^x = y$.

It's easy to calculate the log of powers of 10:

log(10) = 1	10^1 = 10
log(100) = 2	10^2 = 100
log(1000) = 3	10^3 = 100
log(1000000) = 6	10^6 = 1000000

10^0 is defined as 1, so the log of 1 is zero.

For numbers between 1 and 0, logs are negative

$\frac{1}{10} = 10^{-1}$	log(0.1) = -1
$\frac{1}{100} = 10^{-2}$	log(0.01) = -2
$\frac{1}{1000} = 10^{-3}$	log(0.001) = -3

The $\log_{10}$ of powers of 10 are integers, but we can raise 10 to non-integer powers too, to get the log of any number greater than zero. For instance, $10^{2.09}$ is 123, so the log of 123 is 2.09.

We can see with round powers of 10 than using logs we can move between multiplication and addition:

100 × 1000 = 100000 $10^2 \times 10^3 = 10^5 = 10^{2+3}$

Thus do calculate A × B we do as follows:

- Calculate log(A)
- Calclate log(B)
- Calculate log(C) = log(A) + log(B)
- Take the anti-log of log(C), i.e., $10^{\log(C)}$ = C

2 An application

If you have a certain quantity (e.g., money in a bank account), whose value increases by a constant proportion every year, its value in any year depends on a multiplicative relationship. Let's say the increases is α (i.e., a 10% increase means α = 1.1)

Year 0	100
Year 1	$100 \times \alpha$
Year 2	$100 \times \alpha \times \alpha$
Year 3	$100 \times \alpha \times \alpha \times \alpha$
Year 4	$100 \times \alpha \times \alpha \times \alpha \times \alpha$
Year 5	$100 \times \alpha \times \alpha \times \alpha \times \alpha \times \alpha$

In short, the value in year t is $100 \times \alpha^t$

$$y_t = 100 \times \alpha^t$$

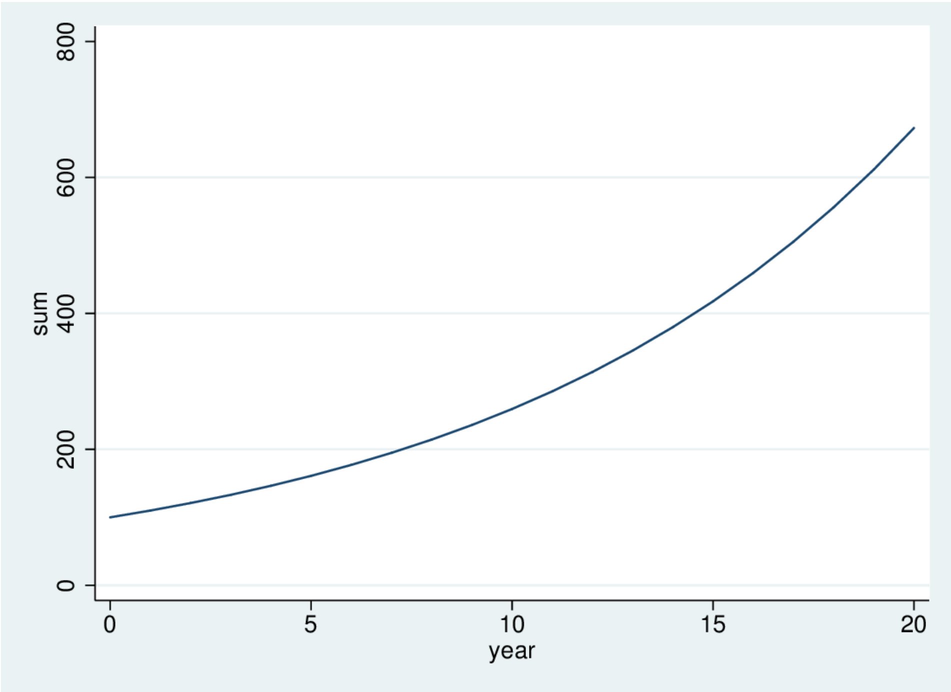

Figure 1: A constant proportional increase

See Fig 1, which shows how the sum increases "exponentially" over time.
But if we convert to logs we can calculate it as follows

$$log(y_t) = log(100) + t \times log(\alpha)$$

In other words, rather than multiplying by α every year, we add log(α).
This gives a straight line relationship (see Fig 2).
Thus we can use logs to move between multiplicative and additive (straight-line) relationships.

3 Other bases

Logs to the base 10 are easy to understand, but the base number need not be 10. A log to the base n is defined thus:

$$y = log_n(x) \Leftrightarrow n^x = y$$

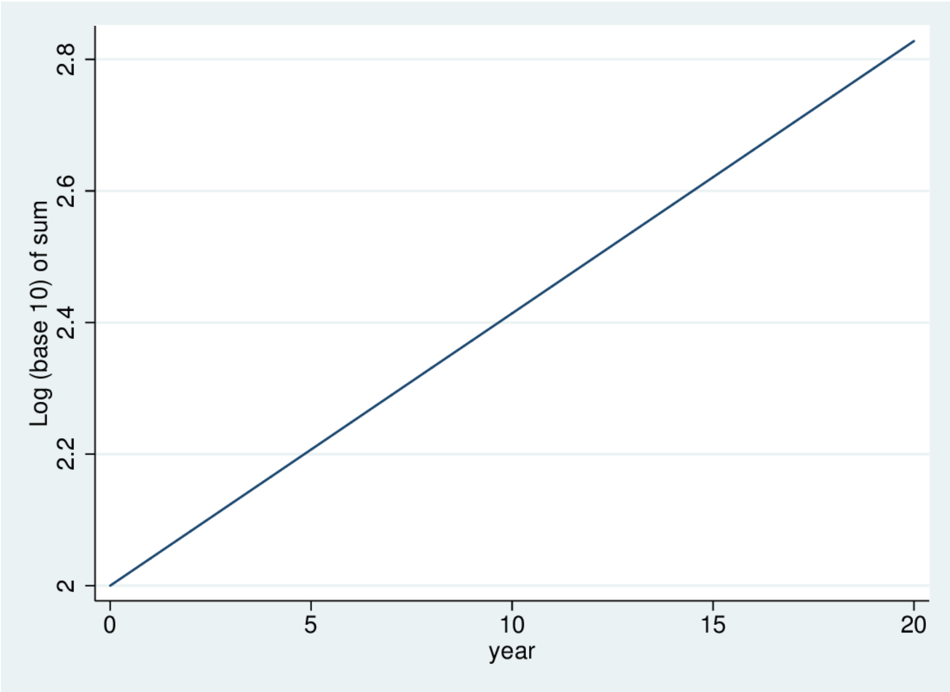

Figure 2: Taking the base-10 log of the sum: a straight line

Computer scientists often use $\log_2$, but the most common log base is the special number $e \approx 2.7183$. This has some special mathematical properties that make certain calculations easier. Logs to base e are called natural logs, often written ln(x) etc:

$$y = ln(x) \Leftrightarrow e^x = y$$

See Fig 3, which shows that the natural log also gives a straight line.

Fig 4 shows the natural log of X from 0.1 (-2.303) to 100 (4.605). For X = 1, the log is 0. As X approaches 0, the log falls faster and faster. As X rises above 1, the log rises, but more slowly as it goes. Note that the log rises from X = 5 to 10 as much as it does from X = 40 to 80.

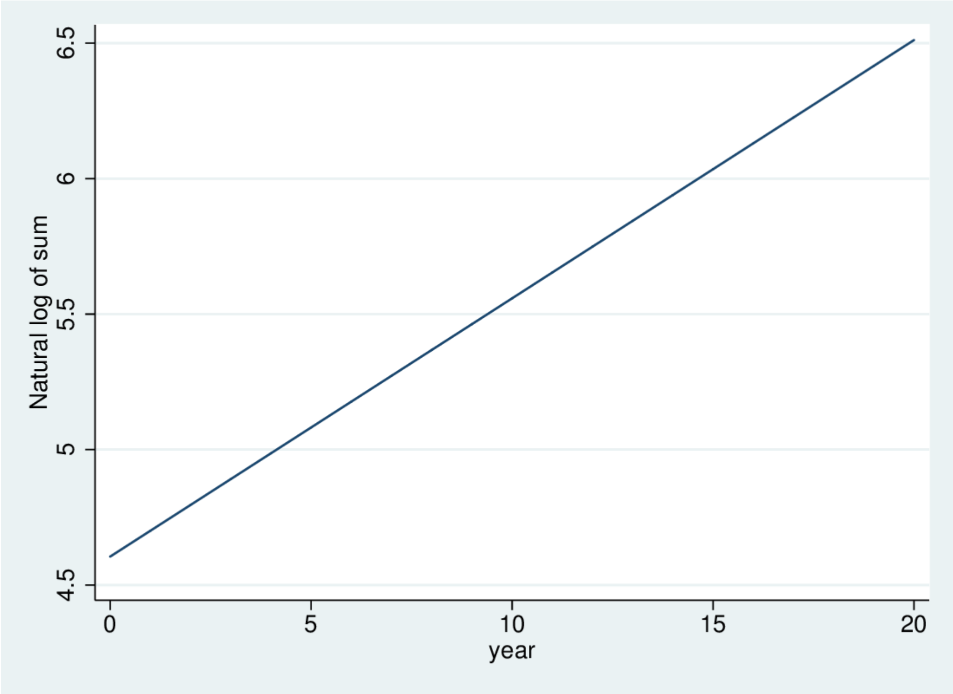

Figure 3: Taking the natural log of the sum: also a straight line

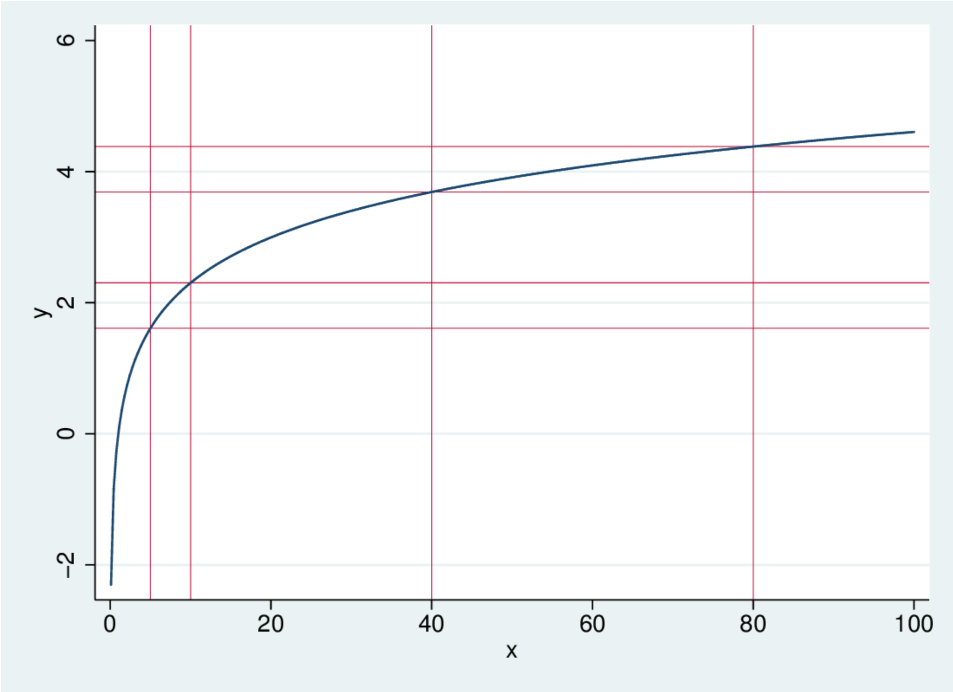

Figure 4: The natural log of X for X from 0.1 to 100