UL Summer School – Refresher Session

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Outline

The Quantitative Method

Sampling

More on statistical tests

What is Quantitative Method?

- Distinct in using number
- Large amounts of relatively shallow data
- Data may be shallow, but is strictly comparable: compare and contrast
- Tends to look to explicit causal explanations

What is Quantitative Method?

- Clearly defined meanings allocated numerical representations
- Thus easily manipulated
- Descriptive statistics and graphics
- Analytical statistics and graphics

Causal relationships from empirical data?

- QM often concerned with causal accounts, “low level” theories
- The experiment is probably the strongest way of arguing from data
  - “Experimental control” means everything is the same except the input of interest
  - A strong inference that differences in the result are caused by the difference in the input
- Experiments are rarely possible in social science: therefore we use “observational” data, and compare and contrast (“statistical control”)

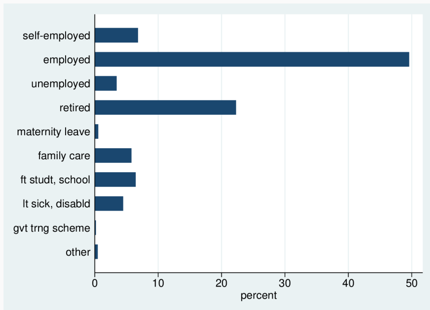
Numbers as information

- Designing – asking structured questions so answers can be mapped onto numbers
- Coding – turning answers to numbers and entering them on the computer
- Labelling – attaching meaning to the numbers (not essential but very helpful!)
- Reporting/analysing – very easy once the preceding steps are accomplished

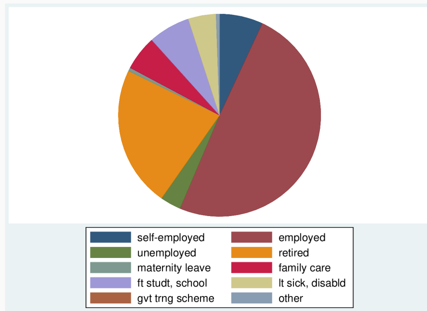
Univariate summaries of data

- Numerical summaries
  - Data type hierarchy: Nominal, Ordinal, Interval, Ratio
  - Measures of central tendency: means, modes and medians
  - For data with few distinct values: frequency distributions
  - Measures of spread: range, inter-quartile range, standard deviation
- Graphical summaries
  - Bar and pie charts
  - Histograms
  - Box plots

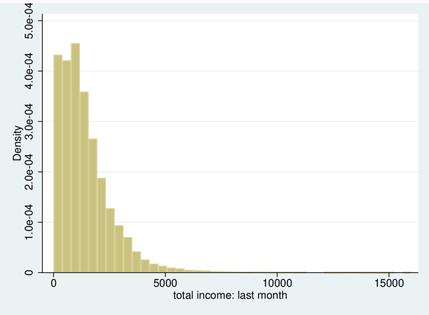
Example 1



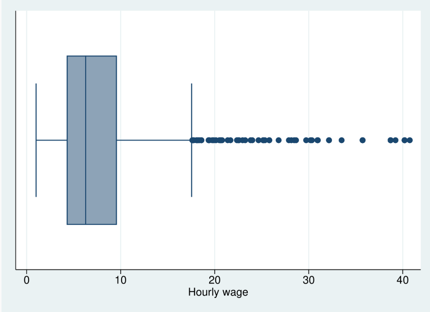
Example 2



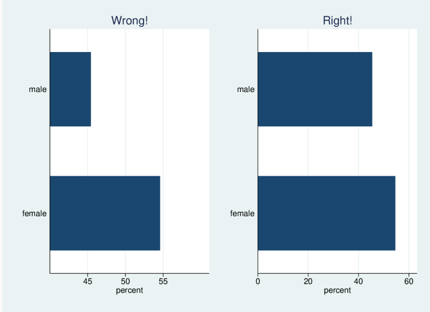
Example 3



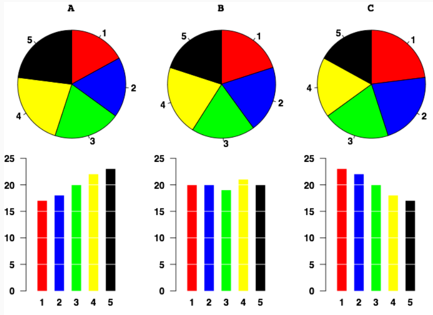
Example 4



Example 5



Example 6



Bivariate summaries of data

- Numerical
  - Crosstabulations
  - Comparing means across groups
  - Correlations
- Graphical
  - Side-by-side and stacked bar charts, histograms, box-plots
  - Scatter plots

Crosstab

```
. tab empstat sex, col
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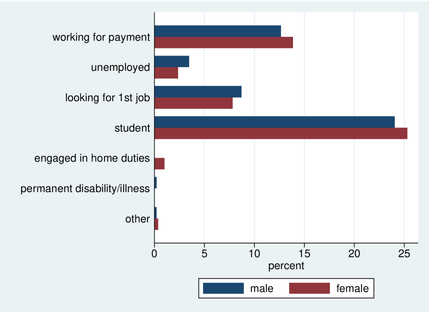
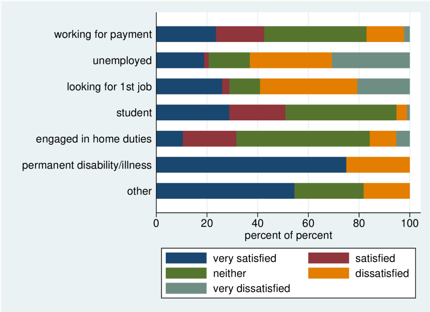
Key: *frequency* / *column percentage*

usual employment situation {s81}	school leavers sex {s11} male	female	Total
working for payment	241 25.67	264 27.33	505 26.51
unemployed	66 7.03	45 4.66	111 5.83
looking for 1st job	166 17.68	149 15.42	315 16.54
student	458 48.78	482 49.90	940 49.34
engaged in home dutie	0 0.00	19 1.97	19 1.00
permanent disability/	4 0.43	0 0.00	4 0.21
other	4 0.43	7 0.72	11 0.58
Total	939 100.00	966 100.00	1,905 100.00

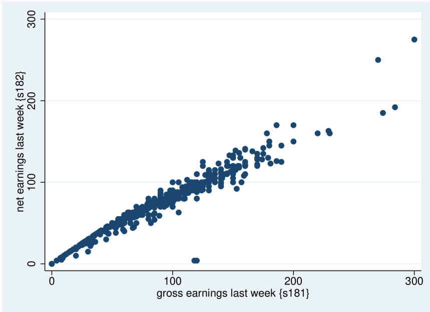
Bivariate Barchart



Proportional barchart



Scatterplot



Sampling

- The use of sampling is another characteristic of QM
- Calculations based on representative samples *approximate* those of the reference population
- *Random* sampling is a powerful way of ensuring representativity
- What does *random* mean?

Simple random sampling

- In “Simple Random Sampling” every element in the reference population has the same chance of being selected
- SRS needs a clear *sampling frame* (e.g., a list of everyone in the population) and a random selection process
- E.g., a list of all students in a university, “put the names in a hat”
- Often difficult to get a good sampling frame
- SRS more important as an ideal type for reasoning about statistics

Varieties of sampling

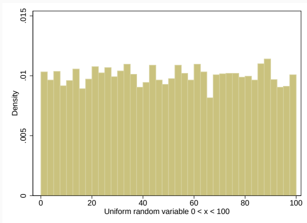
- Non-representative: accidental sampling, volunteer sampling
- Quasi-representative: quota sampling
- Representative: SRS, cluster sampling, stratified sampling
- What is representativity?

Data distributions

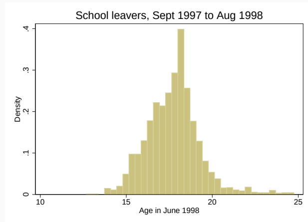
- We have seen how to display and summarise the distribution of variables:
  - Categorical: frequency distribution, percentage distribution, bar and pie charts
  - Continuous (interval/ratio): mean, median, IQR, standard deviation, histogram, box-plot

Uniform distribution

- The shape of the histogram tells us about the distribution of the variable
- If a variable is “uniformly” distributed we see a flat distribution between the extremes:



- More often we see “heaped distributions” where more of the observations cluster around the centre, like this age example from the ESRI School-Leavers’ Survey:

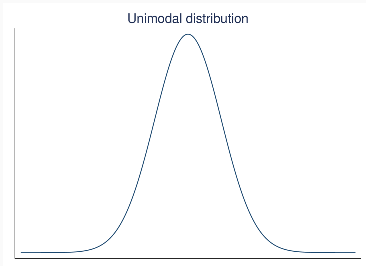


Distribution patterns

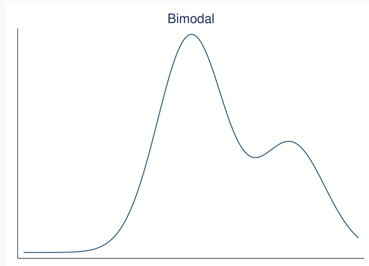
There are many patterns we might see in histograms and distributions:

- Uniform
- Extremes
- Bimodal
- Uni-modal
  - Asymmetric
    - Positively skewed (to right)
    - Negatively skewed (to left)
  - Symmetric (with different levels of **kurtosis**)
    - platykurtic – flatter
    - mesokurtic – average
    - leptokurtic – very concentrated around centre

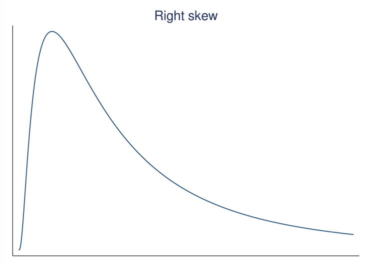
Symmetric unimodal



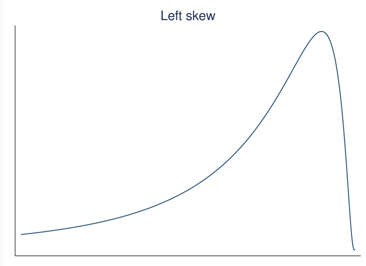
Asymmetric bimodal



Asymmetric: right skew



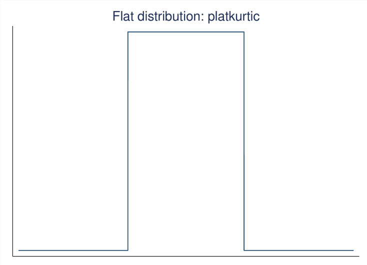
Asymmetric: left skew



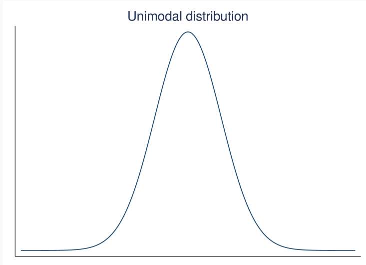
Different symmetric shapes: kurtosis

- Different distributions with the same mean and standard deviation can have different shapes
- Kurtosis: balance between peak, shoulders and tails

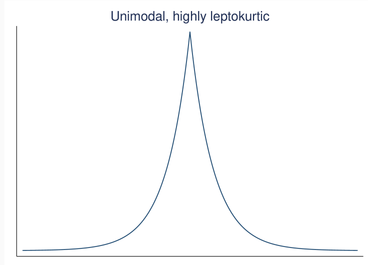
Flat: low kurtosis



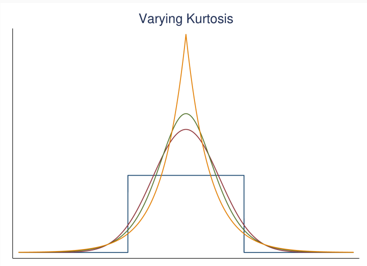
Normal: mid-kurtosis



Very peaky: high-kurtosis



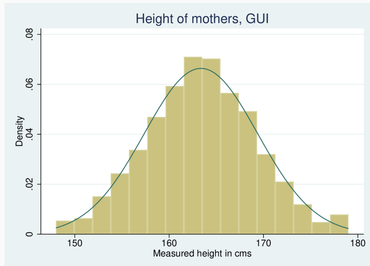
Varying kurtosis



The Normal Distribution

- One very important theoretically defined distribution is the “Normal Distribution”
- Occurs in the real world, e.g., where multiple measures of a fixed value have additive errors, or where there is a typical value around which cases are distributed
- The normal distribution is
  - symmetric (no skew)
  - mesokurtic (between flatter and pointier)
- The mean, mode and median are the same
- The farther you go from the mean, the lower the proportion of cases, in each direction symmetrically

Approx normal distribution of height



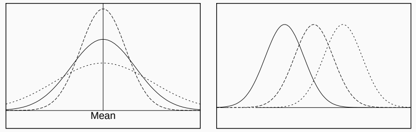
Visualisations

https://commons.wikimedia.org/wiki/File:Galton_box.webm

https://teaching.sociology.ul.ie/so4046/quincunx.mp4

Defined by mean and std deviation

- What makes the normal distribution useful is that its form is well understood:
  - It is completely characterised by its mean and its standard deviation

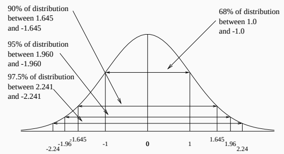


Same mean, different SD　　Same SD, different mean

- http://teaching.sociology.ul.ie:3838/apps/normsd

Reading the Normal Distribution

- About 68% of the cases in a normal distribution will be within 1 standard deviation on either side of its mean
- 95% of cases will be within $\pm$ 1.96 std dev of the mean
- 97.5% of cases will be within $\pm$ 2.24 standard deviations of the mean



- http://teaching.sociology.ul.ie:3838/apps/snd

The most important thing!

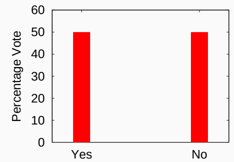
- The immediately most important thing about the normal distribution?
- Take a large sample from a population and calculate a statistic (e.g., a mean)
- Repeat a large number of times and make a histogram of your results
- These will cluster around the true population mean in a normal distribution, with
  - Mean: μ, the true population value
  - Standard deviation: $\frac{\sigma}{\sqrt{N}}$
- $\Rightarrow$ Sample statistics are normally distributed

How wrong are samples?

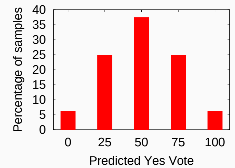
- A random sample gives an "approximately correct" result – how wrong is it likely to be?
- Large samples are more correct, measures of things with more variability are likely less correct
- Explore a simple case:
  - Binary outcome: yes or no (say 50:50 in population)
  - Sample size of 4 (very small, work through the details by hand)

Sampling distribution, N=4

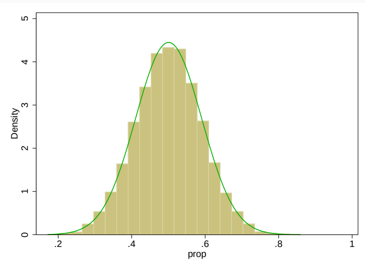
Distribution of original variable



Sampling distribution for N=4



N=1,000; Replications: 10,000



App: Simulate the binomial distribution

http://teaching.sociology.ul.ie:3838/binsim

The Central Limit Theorem

- For a sufficiently large sample, sample estimates are distributed normally
  - Mean: μ, the true population value
  - Standard deviation: $\frac{\sigma}{\sqrt{N}}$
  - The "standard deviation of the sampling distribution" is called the "standard error"
- This holds no matter what the distribution of the original variable
- (Some analyses use other distributions that give better results with smaller samples)

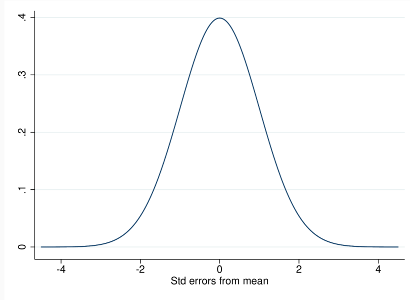
Sampling distributions

- Therefore, any statistic calculated on a single sample can be considered as being drawn from a "sampling distribution" with mean μ and standard deviation $\frac{\sigma}{\sqrt{N}}$
- This allows us to reason about how much sampling error we can expect
- For instance, 95% of the time our sample statistic will be in the range $\mu \pm 1.96 \times \frac{\sigma}{\sqrt{N}}$

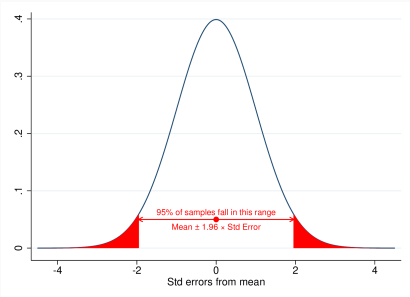
"Confidence" intervals

- If we know that we have a 95% chance of falling in the range $\mu \pm 1.96 \times SE$, we can turn it around:
- There is a 95% chance that the true answer is in the range $\bar{x} \pm 1.96 \times SE$
- Since we don't know σ, the population standard deviation, we estimate it using the sample standard deviation, *s*:
- Confidence interval: $\bar{x} \pm 1.96 \times \frac{s}{\sqrt{N}}$
- Interpretation: in 95% of large simple random samples, the true value will fall within the CI

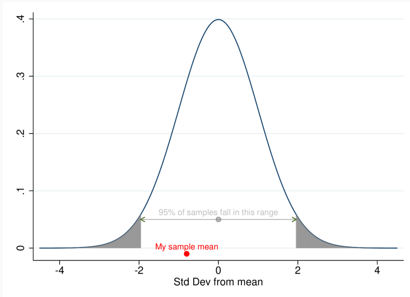
Central limit theorem: sample statistics normal



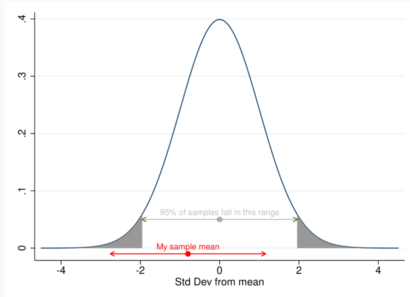
95% within ± 1.96 SE of mean



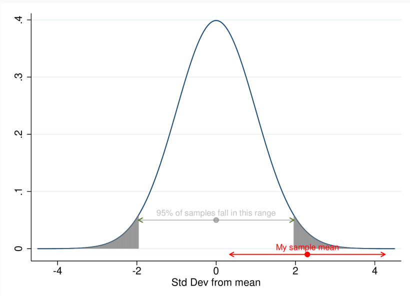
A sample: one of 95% within ± 1.96



Sample mean ± 1.96 SEs contains true mean



But 5% of sample means do not



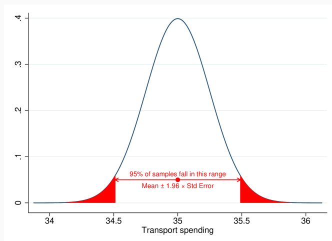
Example: transport spending

- Let's say spending on transport has a true mean of €35 per week, standard deviation €10. With a sample of 1600, 95% of all possible sample estimates will fall between:

$$35 \pm 1.96 \times \frac{10}{\sqrt{1600}}$$

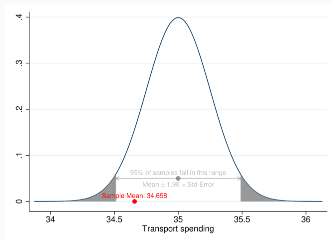
which is

$$\mu \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$



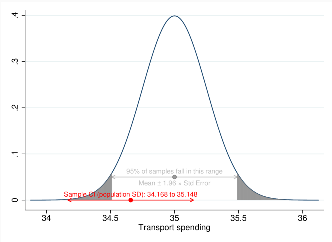
Sample results

- Let's say our sample gives a mean of €34.658, with a standard deviation of €10.123



Reverse the reasoning

- We don't know $\mu = 35$, only $\bar{X} = 34.658$
- We can reverse the reasoning and say that the true value has a 95% chance of falling in the range $\bar{X} \pm 1.96 \times \sigma_{\bar{X}}$



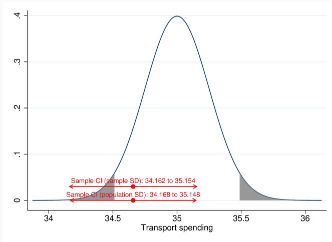
Sample SD

- But we don't know the true standard error
- We can use the sample estimate instead:

$$34.658 \pm 1.96 \times \frac{10.123}{\sqrt{1600}}$$

- In this case, a very slightly wider interval

$$\bar{X} \pm z_{0.95} \times \hat{\sigma}_{\bar{X}}$$



Calculating the interval

- Thus with sample mean is €34.658 and sample standard deviation €10.123 we calculate:
  - Standard Error is $\frac{10.123}{\sqrt{1600}} = \frac{10.123}{40} = 0.2531$
  - The lower bound is $34.658 - 1.96 \times 0.2531 = 34.162$
  - The upper bound is $34.658 + 1.96 \times 0.2531 = 35.154$
- We can interpret this as saying we are 95% confident that the true value is in this range
- This is because with 95% of all possible samples, such a confidence interval will include the true value

Different confidence levels

- 95% implies a 1 in 20 chance of the true value not falling in the interval
- Confidence levels can be changed: 99% CI is $\bar{x} \pm 2.58 \times SE$

CIs and proportions

- We can also calculate CIs around proportions: $p \pm 1.96 \times SE_p$
  - The SD of a proportion p is $\sqrt{p \times (1-p)}$
  - The SE of a proportion is therefore $\sqrt{\frac{p \times (1-p)}{N}}$
- For $N = 1,000$ the CI is $\pm 3\%$:

$$1.96 \times SE = 1.96 \times \sqrt{\frac{0.5 \times 0.5}{1000}} = 0.031$$

Student's t-distribution

- The CLT speaks of large samples
- For smaller samples it is only approximately true, and the CI as defined above will be too narrow
- For means, we can use "Student's t distribution" in place of the normal distribution
- For large samples, we use $z = 1.96$ for 95% CIs; for smaller samples we use $t > 1.96$
- This makes for wider, correct CIs

CIs – how fuzzy is the answer?

- Confidence intervals allow us to present sample information appropriately
  - Point estimate, *e.g.*, mean or other sample statistic: our "best guess" of the true value
  - Confidence interval: range in which we are "confident" true value (the population parameter) lies
- In combination, the point estimate and CI give us the answer with a measure of its precision

Hypothesis testing

- Hypothesis testing is a way of using the reasoning behind CIs to make specific claims about the population
- Say we want to know if there is a relationship between two variables, *e.g.*, whether taking a particular qualification has an effect on wages (positive or negative)
  - We begin with a hypothesis: $W_a \neq W_b$ or $W_a > W_b$
  - We negate the hypothesis to form a "{null hypothesis}", called H_0:

$$H_0 : W_a = W_b$$

$$\Rightarrow H_0 : D_w = W_a - W_b = 0$$

- That is, on average, the wage difference is zero

Testing the null hypothesis

- First we calculate a sample mean wage difference, $\hat{D}_w$
- Then we construct a confidence interval at our chosen level of confidence (*e.g.*, 95%): $\hat{D}_w \pm z_{0.95} \times SE$
- If zero lies outside the interval, we are **at least** 95% sure the true population value is not zero, and we can **reject the null hypothesis**
- If zero lies within the interval, then zero is in the range of plausible values, so we cannot reject the null hypothesis
- We don't say we "accept the null hypothesis" because zero is just in the range of plausible values, and other values in this range are approximately as likely

Rejecting the null hypothesis?

- Rejecting the null hypothesis constitutes support for the initial or "alternative" hypothesis
- Failing to reject the null hypothesis means the data fail to support the initial hypothesis: "there is no evidence that the course affects wage"
- Failure to support the initial hypothesis may be because
  - It is actually false, *i.e.*, $D_w = 0$
  - The effect is small and/or very variable, and thus the sample is too small to detect it

Significance

- Let's say we do a hypothesis test with a 95% confidence level, and we find the zero is way outside the CI
- We can try again with a 99% confidence level:
  - If it is still outside the interval we are not "at least 95%" but "at least 99%" sure that zero is not the true value
- If we keep trying with CIs with higher confidence levels we will eventually find one where zero is just outside the interval
- If that is at confidence level C we can say that we are C% sure (not "at least" any more) that zero is not the true value

p-values

- This $p = 100\% - C\%$ value is the probability that we get a sample statistic as different from zero as we did, even though the true value was zero
- This is known as the significance of the sample estimate, or its p-value
- We want it to be as small as possible, typically under 5% (0.05)
- p-values are widely used – stats programmes report them in many places
- In general the interpretation is "what's the probability of getting this result by chance if the null hypothesis was true?"

t-test shortcuts

- Rather than try repeatedly to get a CI just short of touching zero, we can calculate a t-statistic:

$$t = \frac{\bar{x}}{SE}$$

- If this t is greater than the "critical value", e.g., 1.96 for large samples at 95% confidence, we can reject the null hypothesis
- If the CI doesn't include zero, t will be greater than the critical value
- We can also calculate the exact p-value for the t-statistic

Error in hypothesis testing

- Another way of looking at significance is "the chance we would be wrong if we believe the initial hypothesis"
- For instance, if there is one chance in twenty ($p = 0.05$) that the true value is outside the CI, then by basing our decision on the CI we will be wrong one time in twenty
- This is known as Type I Error: rejecting the null hypothesis when it is true
  - *e.g.*, the true value might be zero but a small number of possible samples generate CIs that don't include zero
- If it is very important to avoid Type I error, we use high confidence levels (*e.g.*, 99.5% instead of 95%) or insist on low p-values (*e.g.*, 0.005 instead of 0.05)

Type II error

- However, there is a second type of error, Type II
- Type II Error is failing to reject the null hypothesis when it is false
- That is, failing to support the initial hypothesis even though it is true
- If we raise the confidence level we reduce the risk of Type I error but raise the risk of Type II error
- That is, if we make a special effort not to accept an initial hypothesis unless there is very clear evidence, we necessarily fail to accept it where there is only fairly clear evidence
- For a given p-value, we can only reduce the Type II error by increasing the sample size

Association in tables

- We detect association in tables many ways
  - Comparing row percentages up and down columns
  - Column percentages across rows
  - Comparing observed with *expected* values
  - "Expected" $\Rightarrow$ the concept of "independence"

The χ^2 test for association in tables

- Independence: no association between two variables
  - pattern of row percentages the same in all rows
  - pattern of column percentages the same in all columns
- But even if independence holds in the population, sampling variability leads to differences in percentages
- How big can the differences be before we can be convinced that there is really association in the population?

Observed and Expected

- Method: compare the real table ("observed") with hypothetical table under independence ("expected")
- Summarise the difference into a single figure (χ^2 statistic, chi-sq)
- Compare χ^2 statistic with known distribution
- ... What is the probability of getting a sample statistic "at least this big" by simple sampling variability *if independence holds in the population*?

Calculating the χ^2 statistic

- The "expected" table has the same row and column totals, but the cell values are such that the percentages are the same as in the total row and column:

$$n_{ij} = \frac{R_i C_j}{T}$$

- For each cell we summarise the difference between observed (O) and expected (E) values as

$$\frac{(O-E)^2}{E}$$

- The summary for the table as a whole is the sum of this quantity across all cells:

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

χ^2 distribution

- This statistic is known to have a predictable distribution, the χ^2 distribution
- That is, if we take a large number of samples from a population where there is no association, and calculate the statistic, they will have a distribution in a known form, and we can calculate the probability of finding a value "at least as large as" any given number
- The distribution depends only on the "degrees of freedom" which is the number of rows minus one times the number of columns minus one:

$$df = (r-1)(c-1)$$

Critical values and hypothesis testing

- Reading the table, we go to the row corresponding to the degrees of freedom, and read across until we get to the column with our chosen probability level (say 0.05) – this gives us the appropriate "critical value"
- If our χ^2 is bigger than the critical value, then there is at most one chance in 20 (*i.e.*, 0.05) that it has arisen by sampling variability and a 95% chance (*i.e.*, $1-0.05$) that it is due to real association in the population
- When using computers, the exact p-value of the calculated χ^2 statistic is reported: if $p \leq 0.05$ then we can reject the null hypothesis of no association with 95% confidence

Critical values for t

- With both the χ^2 test and the t test, we
  - calculate a "test statistic"
  - compare it with a "critical value", and/or
  - calculate its exact p value
- When testing for association in tables, $\chi^2 = \sum \frac{(O-E)^2}{E}$
- When comparing a mean to zero, $t = \frac{\bar{x}}{SE}$

Multiple t-tests

- We have looked at the "paired sample" t-test, where we compare a difference (between paired observations) to zero: $t = \frac{\bar{x}}{SE}$
- This is a special case of the one-sample t-test, where we compare a sample statistic to a fixed reference value, r: $t = \frac{\bar{x}-r}{SE}$
- This also applies to proportions, comparing to a reference value such as 50%:

$$t = (p-r)/\sqrt{p(1-p)/N}$$

(note: use normal distribution, not t-distribution, as long as sample is large enough)

The "independent-sample t-test"

- A third case is the "independent-sample t-test", where we compare means across different (sub-)samples
- If we wish to test for differences across groups (e.g., differences in income between men and women) we are comparing one sample mean with another, not a sample mean with a fixed value
- We can consider the sample difference $(\bar{x}_m - \bar{x}_w)$ to be a point estimate of the population difference
- The null hypothesis is that $\mu_m = \mu_w$, or $\mu_m - \mu_w = 0$

Testing for differences in means

- To construct a CI or calculate the test statistic, we need the SE
- Where the groups have different variances this is

$$\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

- If we can assume the sub-population variances are the same it simplifies to

$$\sqrt{\frac{s^2}{n_1} + \frac{s^2}{n_2}} = s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Test statistic

- The test statistic is standard, using the appropriate SE:

$$t = \frac{x_m - x_w}{SE}$$

- The degrees of freedom are complicated to calculate in the general case
- In the equal-variance case they are $n-2$ as two sample statistics are calculated